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SO(2, 1) and strong Coulomb coupling

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Abstract. We show explicitly, in the framework of the Klein–Gordon equation, that the algebraic method based upon unitary irreducible representations of the group SO(2, 1) used to solve the problem of strong Coulomb coupling ($e^2Z > l + \frac{1}{2}$) is equivalent to constructing solutions that are orthogonal with respect to some mixed scalar product, rather than the standard Klein–Gordon scalar product. This elucidates the difference between the spectra given by Case's method, on the one hand and the algebraic method, on the other hand. By explicitly computing scattering states, we further show that algebraic solutions describe absorption of the particle as in the corresponding classical problem.

1. Introduction

As is well known (Case 1950), the Coulomb potential for a Klein–Gordon (KG) particle or a Dirac particle becomes singular when the coupling constant e^2Z is greater than, respectively $\frac{1}{2}$ or 1. (Throughout this work, Ze signifies the charge of the external Coulomb field. We use units such that $\hbar = c = 1$.) This problem was recently solved (Barut and Bornzin 1971) using an algebraic method based upon unitary irreducible representations (UIR) of the group SO(2, 1). The resulting spectrum in this approach is markedly different from the one derived by Case (1950). In particular, for non-zero values of the arbitrary constant present in both cases, the algebraic method allows for states of zero total energy E only when Z is infinite, whereas, in Case's method, upon crossing $E = 0$, the energy bound state can either become negative or increase again with increasing Z (Case 1950). This is a somewhat paradoxical situation, as both methods purport to do the same thing, i.e. provide a set of solutions defining a domain over which the Klein–Gordon or Dirac Hamiltonian is self-adjoint. In this paper, we solve that paradox by showing, in the framework of the KG equation, that, in contrast to Case's method, the algebraic method does not lead to solutions that are orthogonal with respect to the standard KG scalar product, but rather to a mixed scalar product involving both positive and negative energy bound states. By explicitly computing scattering states, we shall see that this scalar product can be interpreted as describing absorption of the incident particle, in agreement with the behaviour of the solution to the corresponding classical problem. Section 2 shows, for the KG equation, the equivalence between

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the algebraic method and our mixed scalar product approach for getting the bound states spectrum. Section 3 elucidates the meaning of this scalar product by deriving an explicit solution to the problem in the scattering region. Section 4 states our conclusions.

2. Bound states of a Klein–Gordon particle in a strong Coulomb field ($Ze^2 > l+\frac{1}{2}$)

The radial Klein–Gordon equation for a particle of mass m_0 is given by (Case 1950):

$$\frac{d^2u}{dr^2}+\left(-(m_0^2-E^2)+\frac{2Ee^2Z}{r}+\frac{e^4Z^2-l(l+1)}{r^2}\right)u=0. \tag{1}$$

Writing:

$$k=\frac{Ee^2Z}{(m_0^2-E^2)^{1/2}}, \qquad \rho=2(m_0^2-E^2)^{1/2}r, \qquad m=i\lambda, \qquad \lambda=[e^4Z^2-(l+\tfrac{1}{2})^2]^{1/2},$$

(1) becomes

$$\frac{d^2u}{d\rho^2}+\left(-\frac{1}{4}+\frac{k}{\rho}+\frac{\frac{1}{4}-m^2}{\rho^2}\right)u=0. \tag{2}$$

A solution to (2) bounded at infinity is given by (Gradshteyn and Ryzhik 1965)

$$u(k,\lambda,\rho)=CW_{k,i\lambda}(\rho) \tag{3}$$

where $W_{k,i\lambda}(\rho)$ is Whittaker's function, and C is an arbitrary constant. Note that because of the singular nature of the problem (Case 1950), solutions given by (3) are finite at the origin for *all* values of k. Case (1950) then requires solutions to (1) to be orthogonal with respect to the standard KG product:

$$(u_{E_1},u_{E_2})\equiv\int\left(E_1+E_2+\frac{2Ze^2}{r}\right)u_{E_1}^*u_{E_2}\,dr. \tag{4}$$

Let us now introduce the scalar product $\{u_{E_1},u_{E_2}\}$, defined as:

$$\{u_{E_1},u_{E_2}\}=(u_{E_1},u_{E_2})+(u_{-E_1},u_{-E_2}) \tag{5}$$

where (u_{E_1},u_{E_2}) is defined in (4). From (1), we find, in the usual way:

$$(|E_1|-|E_2|)\{u_{E_1},u_{E_2}\}=u_{E_2}u_{E_1}^{*\prime}-u_{E_2}'u_{E_1}^*-u_{-E_2}u_{-E_1}^{*\prime}+u_{-E_2}'u_{-E_1}^*\Big|_0^\infty. \tag{6}$$

Using the formula (Gradshteyn and Ryzhik 1965):

$$W_{k,i\lambda}(\rho)\underset{\rho\to0}{\simeq}\frac{\Gamma(-2i\lambda)\rho^{i\lambda+\frac{1}{2}}}{\Gamma(\frac{1}{2}-i\lambda-k)}+\frac{\Gamma(2i\lambda)\rho^{\frac{1}{2}-i\lambda}}{\Gamma(\frac{1}{2}+i\lambda-k)} \tag{7}$$

we find from (6), taking $C=2^{-1/2}(m_0^2-E^2)^{-1/2}/\Gamma(\frac{1}{2}+i\lambda+k)$ in (3) and performing some straightforward but lengthy algebraic calculations:

$$\{u_{E_1},u_{E_2}\}=\frac{\sin(k_2-k_1)\pi}{|E_2|-|E_1|}. \tag{8}$$

Our solutions will now be orthogonal with respect to the scalar product $\{u_{E_1}, u_{E_2}\}$ if and only if:

$$k = k_0 + n, \tag{9}$$

or:

$$E = m_0\left(1+\frac{e^4Z^2}{(n+k_0)^2}\right)^{-1/2} \tag{9'}$$

with

$$n = 0, \pm 1, \pm 2 \ldots,$$

and k_0 can be restricted, without loss of generality, to be such that $|k_0| < \frac{1}{2}$.

Spectrum (9') is the same as the one derived by Barut and Bornzin (1971), using an algebraic method based upon UIR of the group SO(2, 1). The form of our scalar product (5) was suggested by the work of Montgomery and O'Raifertaigh (1974), showing how to construct IUR of the group SO(2, 1) by means of Whittaker's functions. (One can show in the same way as Montgomery and O'Raifertaigh (1974), that for $e^2Z < l + \frac{1}{2}$ the scalar product (5) reduces to the standard form (4).) It is now clear that Case's method on the one hand and the algebraic method on the other hand do not construct self-adjoint extensions for the same self-adjoint problem. While Case constructs self-adjoint extensions for the KG Hamiltonian defined with the usual KG metric, the algebraic method constructs self-adjoint extensions of the KG Hamiltonian with the metric defined by the scalar product (5). We expect different choices of metrics to lead to different physical pictures, and we shall see that this is indeed the case. While it is known (Rein 1969, Popov 1971) that the arbitrary constant present in Case's solution can be put in a one-to-one correspondence with a cut-off parameter of the cut-off Coulomb problem, we shall show in the next section that the algebraic method leads to a solution describing absorption of the incoming particle in analogy with the corresponding classical problem. Support for this viewpoint can be already found in Barut and Beker (1974), where it is shown how spectrum (9') can be derived from a heuristic quantization of angular momentum in the classical equations.

3. Scattering states

In the scattering region we write the most general solution to (1) in the form (now, $k = Ee^2Z/(E^2 - m_0^2)^{1/2}$):

$$u(k, \lambda, \rho) = A\frac{W_{ik,i\lambda}(-i\rho)}{\Gamma(\frac{1}{2} - i\lambda + ik)} + B\frac{W_{-ik,i\lambda}(i\rho)}{\Gamma(\frac{1}{2} - i\lambda - ik)}. \tag{10}$$

It would be quite tedious to compute phase shifts with (10) using (6), as we did for bound states. Let us note that (6) will be satisfied at $r = 0$ if the conditions:

$$u_E(0) = u_{-E}(0), \tag{11}$$

$$u'_E(0) = u'_{-E}(0) \tag{12}$$

are satisfied. As can be checked, (12) is always satisfied if (11) is, and, furthermore selects the value $k_0 = 0$ in (9). We shall only compute phase shifts for this particular value of k_0, as our main goal is to study what kind of physical picture emerges from the

algebraic approach. For the boundary condition at infinity, we shall require, as is usual for a Coulomb problem:

$$u(k, \lambda, \rho) \underset{r\to\infty}{\to} C(\lambda, k) \sin(pr + k \ln \rho + \delta(\lambda, k)) \tag{13}$$

where $p = (E^2 - m_0^2)^{1/2}$, and $\delta(\lambda, k)$ is the phase shift, C being an arbitrary normalization constant. Using the formula (Gradstein and Ryzhik 1965):

$$W_{k,i\lambda}(\rho) \underset{\rho\to\infty}{\to} e^{-\rho/2}\rho^k, \tag{14}$$

we eventually find, from (7), (11), (13), (14):

$$S(\lambda, k) \equiv \exp(2i\delta(\lambda, k)) = e^{-\lambda\pi}\left(\frac{\cosh \pi(\lambda - k)}{\cosh \pi(\lambda + k)}\right)^{1/2} e^{i\pi/2}\, e^{i(\beta_+ - \beta_-)}, \tag{15}$$

where

$$\beta_\pm = \arg \Gamma(\tfrac{1}{2} - i\lambda \pm ik). \tag{16}$$

Thus we find that the phase shift becomes complex for $e^2Z > l + \frac{1}{2}$, indicating that one absorption ($|S| < 1$) channel has now opened. This is not surprising if one remembers that, in the corresponding classical problem (Cawley 1967, Barut and Beker 1974), the particle is absorbed by the centre of the field for $e^2Z > M$, where M is the classical angular momentum. As, in quantum mechanics, elastic scattering is always present even in the case of total absorption (Gottfried 1966), we get, for the quantum mechanical problem, a complex phase shift describing elastic scattering of the charged particle in the presence of some absorption channel. Our analysis of scattering states thus further supports the claim (Barut and Beker 1974) that the algebraic solution is the quantum mechanical analogue of the classical solution.

4. Conclusions

Our main goal in this paper was to find out why two mathematical methods both purporting to solve the singular Coulomb problem by providing a set of solutions defining a domain over which the Hamiltonian is essentially self-adjoint failed to give the same spectrum. We showed that the origin of this paradox lies in the fact that the scalar product was different in both approaches. We also showed that the phase shifts for the solutions given by the algebraic method become complex for $e^2Z > l + \frac{1}{2}$, in agreement with the corresponding analysis of the classical problem, where, for $e^2Z > M$, the incident particle is absorbed by the centre of the field. The physical picture underlying the algebraic approach is thus essentially different from the one underlying Case's method, which is equivalent to solving the Klein–Gordon equation with a cut-off Coulomb potential, the cut-off radius being much smaller than the particle Compton wavelength.

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